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OUTP-03-02P
hep-th/0301147
January 2003

Yukawa couplings in $SO(10)$ heterotic M-theory vacua

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Abstract

We demonstrate the existence of a class of $\mathcal{N} = 1$ supersymmetric nonperturbative vacua of Hořava-Witten M-theory compactified on a torus fibered Calabi-Yau 3-fold Z with first homotopy group $\pi_1(Z) = \mathbb{Z}_2$, having the following properties: 1) $SO(10)$ grand unification group, 2) net number of three generations of chiral fermions in the observable sector, and 3) potentially viable matter Yukawa couplings. These vacua correspond to semistable holomorphic vector bundles V_Z over Z having structure group $SU(4)_{\mathbb{C}}$, and generically contain M5-branes in the bulk space. The nontrivial first homotopy group allows Wilson line breaking of the $SO(10)$ symmetry. Additionally, we propose how the 11-dimensional Hořava-Witten M-theory framework may be used to extend the perturbative calculation of the top quark Yukawa coupling in the realistic free-fermionic models to the nonperturbative regime. The basic argument being that the relevant coupling couples twisted-twisted-untwisted states and can be calculated at the level of the $Z_2 \times Z_2$ orbifold without resorting to the full three generation models.

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Contents

1 Introduction

The five self-consistent 10-dimensional superstring theories are different vacua of a single underlying 11-dimensional quantum theory, *M-theory*, which has 11-dimensional supergravity as its low energy limit [1]. While the complete formulation of M-theory is not known, a web of perturbative and nonperturbative dualities has been established which connects the different M-theory limits. These dualities provide insight into the nonperturbative behavior of the superstring theories.

One such duality, proposed by Hořava and Witten, connects M-theory compactified on an orbifold $S^1/\mathbb{Z}_2$ with the strong coupling limit of the $E_8 \times E_8$ heterotic string [2]. A class of $E_8 \times E_8$ heterotic string models are the *realistic free-fermionic models* [3]. A remarkable achievement of the realistic free-fermionic models is their successful prediction of the top quark mass [4].

The first goal of this paper is to demonstrate the existence of a class of $\mathcal{N} = 1$ supersymmetric nonperturbative vacua of Hořava-Witten M-theory compactified on a torus fibered Calabi-Yau 3-fold Z with first homotopy group $\pi_1(Z) = \mathbb{Z}_2$, having the following properties:

1. $SO(10)$ grand unification group.

2. Net number of generations $N_{gen} = 3$ of chiral fermions in the observable sector.

3. Potentially viable matter Yukawa couplings.

The nontrivial first homotopy group allows Wilson line breaking of the grand unification group. Our second goal is to discuss how the 11-dimensional Hořava-Witten M-theory framework may be used to extend the perturbative calculation of the top quark Yukawa coupling in the realistic free-fermionic models to the nonperturbative regime.

The tools needed to achieve the above goals have been recently developed. Donagi, Ovrut, Pantev and Waldram [5, 6] presented rules for constructing a class of $\mathcal{N} = 1$ supersymmetric nonperturbative vacua of Hořava-Witten M-theory compactified on a torus fibered Calabi-Yau 3-fold Z with first homotopy group $\pi_1(Z) = \mathbb{Z}_2$, having grand unification group $H = E_6$ or $H = SU(5)$ and net number of generations $N_{gen} = 3$ of chiral fermions in the observable sector. The case with grand unification group $H = SO(10)$ was studied in [7], where the overlap with the free-fermionic models was discussed. The vacua with $H = E_6$, $SO(10)$, or $SU(5)$ correspond to semistable holomorphic vector bundles V_Z over Z having structure group $G_{\mathbb{C}} = SU(3)_{\mathbb{C}}$, $SU(4)_{\mathbb{C}}$, or $SU(5)_{\mathbb{C}}$, respectively, and generically contain M5-branes in the bulk space. Arnowitt and Dutta [8] argue that phenomenologically viable matter Yukawa couplings can be obtained by requiring vanishing instanton charges on the observable orbifold fixed plane and clustering of the M5-branes near the hidden orbifold fixed plane, and provide an explicit $H = SU(5)$ example.

To aid in achieving our first goal of obtaining vacua with $H = SO(10)$, $N_{gen} = 3$, and potentially viable matter Yukawa couplings, we combine the $G_{\mathbb{C}} = SU(4)_{\mathbb{C}}$ rules discussed in [7] with the constraint of vanishing instanton charges on the observable orbifold fixed plane. Instead of restricting ourselves to a set of sufficient (but not necessary) constraints on the vector bundles (as was done in [5, 6, 7, 8]), we consider the most general case. Indeed, the vacua obtained in Section 5 require this generalization.

The key to achieving our second goal is to utilize the correspondence between the free-fermionic models and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification [9]. In the free fermionic models the top quark mass term arises from a twisted–twisted–untwisted Yukawa coupling, at a fixed point in the moduli space.

We can calculate this coupling in the three generation model, or at the level of the (51,3) or (27,3) $Z_2 \times Z_2$ orbifold. While we do not know the precise geometrical realization of the three generation models, the geometry of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is more readily identified. Furthermore, the calculation can be done as either a $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ $SO(10)$ coupling, or a $\mathbf{27}^3$ E_6 coupling. At the free-fermionic point in the moduli space the numerical results will be identical. Thus, to extend the calculation of the top quark Yukawa coupling in the realistic free-fermionic models to the nonperturbative regime, one can compactify Hořava-Witten M-theory on a Calabi-Yau 3-fold which corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. One can choose $SU(n)_{\mathbb{C}}$ vector bundles with $n = 3$ or $n = 4$, corresponding to the E_6 or $SO(10)$ grand unification group, respectively. The nonperturbative top quark Yukawa coupling at the grand unification scale is then computed, at least in principle, using (2.21). We note that the compactification on the Calabi-Yau 3-fold corresponding to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold will lead to $N_{gen} \neq 3$.

We are thus led to present rules for arbitrary N_{gen}, allowing general $SU(n)_{\mathbb{C}}$ vector bundles with $n = 3$, 4, or 5 corresponding to grand unification group E_6, $SO(10)$, or $SU(5)$, respectively. To obtain potentially viable Yukawa couplings, we require vanishing instanton charges on the observable orbifold fixed plane. This further restricts the allowed vector bundles.

This paper is organized as follows. In Section 2, we briefly review Hořava-Witten M-theory, some results of its compactification to four dimensions and the associated 4-dimensional low energy effective theory. In Section 3, we present the rules as discussed above. Section 4 demonstrates that torus fibered Calabi-Yau 3-folds with $\pi_1(Z) = \mathbb{Z}_2$ and a Hirzebruch base surface do not admit the $H = SO(10)$, $N_{gen} = 3$ vacua with potentially viable matter Yukawa couplings that we seek. In contrast, Section 5 demonstrates the existence of such vacua with a del Pezzo dP_7 base. In Section 6, we discuss the extension of the top quark Yukawa coupling calculation in the realistic free-fermionic models to the nonperturbative regime, and explain why modifications to the rules of Section 3 may be required for a detailed analysis. Section 7 summarizes our conclusions.

2 Review of Horava-Witten M-theory

Hořava and Witten proposed that M-theory compactified on an orbifold $S^1/\mathbb{Z}_2$ is the strong coupling limit of the $E_8 \times E_8$ heterotic string [2].

The low energy effective action of Hořava-Witten M-theory can be formulated as an expansion in powers of the 11-dimensional gravitational coupling κ. To lowest order in this expansion, Hořava-Witten M-theory is 11-dimensional supergravity [10], which is of order κ^{-2}. The supergravity vacuum is specified by the metric g_{IJ}, the 3-form potential C_{IJK} with 4-form field strength $G_{IJKL} = 24\partial_{[I}C_{JKL]}$, and the spin 3/2 gravitino ψ_I.

The $\mathbb{Z}_2$ projection introduces gauge, gravitational, and mixed anomalies into the theory. Cancellation of the irreducible part of the gravitational anomaly requires the introduction of two chiral $\mathcal{N} = 1$ E_8 vector supermultiplets, one on each orbifold fixed plane M_i^{10} $(i = 1, 2)$ at $x^{11} = 0$ and $x^{11} = \pi\rho$, respectively. The reducible portion of the gravitational anomaly, as well as the gauge and mixed anomalies, are cancelled with a refinement [2] of the standard *Green-Schwarz mechanism* [11]. Requiring the one-loop chiral anomaly to cancel fixes a relation between κ and the 10-dimensional gauge coupling λ:

$$\frac{1}{\lambda^2} = \frac{1}{2\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}. \tag{2.1}$$

Formally, the low energy effective action of Hořava-Witten M-theory appears to be an expansion with the m^{th} term being of order $\kappa^{-2+(2m/3)}$ $(m = 0, 1, 2...)$. Other exponents must arise at the quantum level since we will run into infinities which, when cut off in the quantum theory, must on dimensional grounds give anomalous powers of κ.

2.1 Compactification to four dimensions

We refer to the compactification of Hořava-Witten M-theory to lower dimensions as *heterotic M-theory*. The compactification to four dimensions with unbroken $\mathcal{N} = 1$ supersymmetry was discussed in [12]. The procedure starts with the spacetime structure

$$M^{11} = M^4 \times Z \times S^1/\mathbb{Z}_2, \tag{2.2}$$

where M^4 is 4-dimensional Minkowski space and Z is a Calabi-Yau 3-fold. M5-branes can be included in the bulk space at points throughout the orbifold interval. These M5-branes are required to span M^4 (to preserve $(3+1)$-dimensional Poincaré invariance) and wrap holomorphic curves in Z (to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions).

Generally, some subgroup G of the E_8 symmetry will survive this compactification. E_8 is broken to $G \times H$, where the grand unification group H is the commutant subgroup of G in E_8. The gauge fields associated with G 'live' on the Calabi-Yau 3-fold, and hence $(3+1)$ Poincaré invariance is left unbroken. The requirement of unbroken $\mathcal{N}=1$ supersymmetry implies that the corresponding field strengths must satisfy the *Hermitian Yang-Mills constraints* $F_{ab} = F_{\overline{a}\overline{b}} = g^{a\overline{b}} F_{a\overline{b}} = 0$. Donaldson [16] and Uhlenbeck and Yau [17] prove that each solution to the 6-dimensional *Hermitian Yang-Mills equations* $D^A F_{AB} = 0$ satisfying the Hermitian Yang-Mills constraints corresponds to a semistable holomorphic vector bundle over the Calabi-Yau 3-fold with structure group being the complexification $G_{\mathbb{C}}$ of the group G, and conversely.

The correction to the background (2.2) is computed perturbatively. The set of equations to be solved consists of the Killing spinor equation

$$\delta\psi_I = D_I\eta + \frac{\sqrt{2}}{288}(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM})G^{JKLM}\eta = 0, \tag{2.3}$$

the equation motion

$$D_I G^{IJKL} = 0 \tag{2.4}$$

and the Bianchi identity

$$\begin{aligned}(dG)_{11RSTU} = 4\sqrt{2}\pi \left(\frac{\kappa}{4\pi}\right)^{2/3} \Bigg[& J^{(0)}\delta(x^{11}) + J^{(N+1)}\delta(x^{11} - \pi\rho) \\ & + \frac{1}{2}\sum_{n=1}^{N} J^{(n)} \left(\delta(x^{11} - x_n) + \delta(x^{11} + x_n)\right)\Bigg]_{RSTU}\end{aligned} \tag{2.5}$$

where $J^{(0)}$, $J^{(N+1)}$ are the sources on the orbifold fixed planes at $x^{11} = 0$ and $x^{11} = \pi\rho$, respectively, and $J^{(n)}$ $(n = 1, \ldots, N)$ are the M5-brane sources located at $x^{11} = x_1, \ldots, x_N$ $(0 \leq x_1 \leq \ldots \leq x_N \leq \pi\rho)$. Note that each M5-brane at $x = x_n$ has to be paired with a mirror M5-brane at $x = -x_n$ with the same source since the Bianchi identity must be even under the $\mathbb{Z}_2$ symmetry.

The Bianchi identity (2.5) can be viewed as an expansion in powers of $\kappa^{2/3}$. To linear order in $\kappa^{2/3}$, the solution to the Killing spinor equation,

equation of motion, and Bianchi identity takes the form

$$(ds)^2 = (1+b)\eta_{\mu\nu}dx^\mu dx^\nu + (g_{AB}^{(CY)} + h_{AB})dx^A dx^B + (1+\gamma)\left(dx^{11}\right)^2 \tag{2.6}$$

$$G_{ABCD} = G^{(1)}_{ABCD} \tag{2.7}$$

$$G_{ABC11} = G^{(1)}_{ABC11} \tag{2.8}$$

$$\eta = (1+\psi)\eta^{(CY)}. \tag{2.9}$$

with all other components of G_{IJKL} vanishing. $g_{AB}^{(CY)}$ and $\eta^{(CY)}$ are the Ricci-flat metric and the covariantly constant spinor on the Calabi-Yau 3-fold.

As discussed in [13], the first order corrections b, h_{AB}, γ, $G^{(1)}$ and ψ can be expressed in terms of a single $(1,1)$-form $\mathcal{B}_{a\bar{b}}$ on the Calabi-Yau 3-fold. All that remains then is to determine $\mathcal{B}_{a\bar{b}}$, which can be expanded in terms of eigenmodes of the Laplacian on the Calabi-Yau 3-fold. For the purpose of computing low energy effective actions, it is sufficient to keep only the zero-eigenvalue or 'massless' terms in this expansion; that is, the terms proportional to the harmonic $(1,1)$ forms of the Calabi-Yau 3-fold. Let us choose a basis $\{\omega_{ia\bar{b}}\}$ for these harmonic $(1,1)$-forms, where $i = 1, \ldots, h^{(1,1)}$. We then write

$$\mathcal{B}_{a\bar{b}} = \sum_i b_i \omega^i_{a\bar{b}} + (\text{massive terms}). \tag{2.10}$$

The $\omega_{ia\bar{b}}$ are Poincaré dual to the 4-cycles $\mathcal{C}_{4i}$, and one can define the integer charges

$$\beta_i^{(n)} = \int_{\mathcal{C}_{4i}} J^{(n)}, \quad n = 0, 1, \ldots, N, N+1. \tag{2.11}$$

$\beta_i^{(0)}$ and $\beta_i^{(N+1)}$ are the instanton charges on the orbifold fixed planes and $\beta_i^{(n)}$, $n = 1, \ldots, N$ are the the magnetic charges of the M5-branes. The expansion coefficients b_i are found in [14] in terms of these charges, the normalized orbifold coordinates

$$z = \frac{x^{11}}{\pi\rho}, \quad z_n = \frac{x_n}{\pi\rho} \quad (n = 1, \ldots N), \quad z_0 = 0, \quad z_1 = 1 \tag{2.12}$$

and the expansion parameter

$$\epsilon = \left(\frac{\kappa}{4\pi}\right)^{2/3} \frac{2\pi^2\rho}{\mathcal{V}^{2/3}}, \tag{2.13}$$

where $\mathcal{V} = \int_Z d^6x \sqrt{g^{(CY)}}$ is the Calabi-Yau volume.

Finally, we note that a cohomological constraint on the Calabi-Yau 3-fold, the gauge bundles, and the M5-branes can be found by integrating the Bianchi identity over a 5-cycle which spans the orbifold interval together with an arbitrary 4-cycle $\mathcal{C}_4$ in the Calabi-Yau 3-fold. Since dG is exact and the cycle is compact, this integral must vanish and we obtain

$$[W_Z] = c_2(TZ) - c_2(V_{Z1}) - c_2(V_{Z2}) \tag{2.14}$$

where $c_2(TZ)$ and $c_2(V_{Zi})$ are the second Chern classes of the tangent bundle TZ and the vector bundle V_{Zi}, respectively and $[W_Z]$ is the 4-form cohomology class associated with the M5-branes.

2.2 Four-dimensional low energy effective theory

Following [15], we now discuss the 4-dimensional low energy effective theory on the observable orbifold fixed plane at $x^{11} = 0$. As discussed in Section 2.1, the apriori E_8 gauge symmetry is broken to $G \times H$. The **248** of E_8 decomposes under $G \times H$ as $\mathbf{248}_{E_8} \to \oplus_{\mathcal{S},\mathcal{R}}(\mathcal{S},\mathcal{R})$, where $\mathcal{S}$ and $\mathcal{R}$ are irreducible representations of G and H, with representation indices $x, y, \ldots = 1, \ldots, \dim(\mathcal{S})$ and $p, q, \ldots = 1, \ldots, \dim(\mathcal{R})$, respectively. We denote a physical field in the representation $\mathcal{R}$ of H by $C^{Ip}(\mathcal{R})$. Here $I, J, K, \ldots = 1, \ldots, \dim\left(H^1(Z, V_{Z1\mathcal{S}})\right)$ is the generation index, $V_{Z1\mathcal{S}}$ is the vector bundle V_{Z1} in the representation $\mathcal{S}$, and the cohomology group $H^1(Z, V_{Z1\mathcal{S}})$ has basis $\{u_I^x\}$.

Define the conventional 4-dimensional chiral fields S, T^i and the chiral fields Z_n by

$$\mathrm{Re}(S) = V; \quad \mathrm{Re}(T^i) = Ra^i; \quad \mathrm{Re}(Z_n) = z_n \tag{2.15}$$

Here $V = \mathcal{V}/v$ where $v = \int_Z d^6x$. The nonvanishing components of the Calabi-Yau metric are given by $g^{(CY)}_{a\bar{b}} = g^{(CY)}_{\bar{b}a} = ia^i\omega_{ia\bar{b}}$ $(i = 1, \ldots, h^{(1,1)})$, where a^i are the $(1,1)$ moduli of the Calabi-Yau 3-fold. The modulus R is the Calabi-Yau averaged orbifold radius divided by ρ, and the moduli z_n $(n = 0, 1, \ldots, N, N+1)$ are given by (2.12).

The 4-dimensional low energy effective theory on the observable orbifold fixed plane is specified in terms of 3 functions of the chiral matter multiplets:

1. The *Kähler potential* $K_{matter} = Z_{IJ}\overline{C}^I C^J$ determines the kinetic terms of the chiral matter fields. To first order in the expansion parameter ϵ, the Kähler metric Z_{IJ} takes the form
$$Z_{IJ} = e^{-K_T/3}\left[G_{IJ} - \frac{\epsilon}{2\mathcal{V}}\tilde{\Gamma}^i_{IJ}\sum_{n=0}^{N+1}(1-z_n)^2\beta_i^{(n)}\right], \tag{2.16}$$
where
$$G^{(\mathcal{R})}_{IJ} = \frac{1}{\mathcal{V}}\int_Z \sqrt{g^{(CY)}}g^{(CY)a\bar{b}}u_{Iax}(\mathcal{R})u^x_{J\bar{b}}(\mathcal{R}) \tag{2.17}$$
$$\tilde{\Gamma}^i_{IJ} = \Gamma^i_{IJ} - (T^i+\overline{T}^i)G_{IJ} - \frac{2}{3}(T^i+\overline{T}^i)(T^k+\overline{T}^k)K_{Tkj}\Gamma^j_{IJ} \tag{2.18}$$
$$K_T = -\ln\left[\frac{1}{6}d_{ijk}(T^i+\overline{T}^i)(T^j+\overline{T}^j)(T^k+\overline{T}^k)\right] \tag{2.19}$$
and
$$K_{Tij} = \frac{\partial^2 K_T}{\partial T^i \partial \overline{T}^j}; \quad \Gamma^i_{IJ} = K_T^{ij}\frac{\partial G_{IJ}}{\partial T^j}; \quad d_{ijk} = \int_Z \omega_i \wedge \omega_j \wedge \omega_k. \tag{2.20}$$

2. The holomorphic *superpotential* W determines the Yukawa couplings
$$Y^{(\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3)}_{IJK} = 2\sqrt{2\pi\alpha_G}\int_Z \Omega\wedge u^x_I(\mathcal{R}_1)\wedge u^y_J(\mathcal{R}_2)\wedge u^z_K(\mathcal{R}_3)f^{(\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3)}_{xyz} \tag{2.21}$$
as well as the F-term part of the scalar potential. Ω is the covariantly constant $(3,0)$ form and $f^{(\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3)}_{xyz}$ projects out the singlet in $\mathcal{R}_1\times\mathcal{R}_2\times\mathcal{R}_3$ (if any). The Yukawa contribution to the superpotential is
$$W_Y = e^{K_{mod}/2}\frac{1}{3}Y_{IJK}C^I C^J C^K, \tag{2.22}$$
where $K_{mod} = -\ln(S+\overline{S}) + K_T$ is the moduli contribution to the Kähler potential.

3. The holomorphic *gauge kinetic function*
$$f = S + \epsilon T^i\left[\beta_i^{(0)}\sum_{n=1}^{N}(1-Z_n)^2\beta_i^{(n)}\right] \tag{2.23}$$
determines the gauge kinetic terms and contributes to the gaugino masses and the gauge part of the scalar potential.

We note that (2.16) and (2.21) hold at the grand unification scale M_G, which coincides with the compactification scale $\mathcal{V}^{1/6}$. The fermion mass hierarchies are encoded in the Kähler metric, which must be diagonalized and rescaled to the unit matrix to obtain the Yukawa couplings of the canonically normalized fields [8]. One then uses the supersymmetry renormalization group equations to evaluate the Yukawa couplings at low energy.

If the perturbative correction to the background discussed in Section 2.1 is to make sense, the second term in (2.16) must be a small correction to the first. However, setting $\mathcal{V}^{1/6} = M_G = 3 \times 10^{16}$ GeV, one finds $\epsilon \simeq 0.93$. Furthermore, one expects G_{IJ} and $\tilde{\Gamma}_{IJ}$ to be of order 1. Arnowitt and Dutta [8] point out that the second term can still be a small correction to the first if the instanton charges on the observable orbifold fixed plane (at $x^{11} = 0$) vanish and the M5-branes cluster near the hidden orbifold fixed plane (at $x^{11} = \pi\rho$):

$$\beta_i^{(0)} = 0 \tag{2.24}$$

$$d_n \equiv (1 - z_n) \ll 1, \quad n = 1, \ldots, N. \tag{2.25}$$

We will impose the $\beta_i^{(0)} = 0$ constraint in Section 3.

3 Summary of rules

In this section, we present rules for constructing a class of $\mathcal{N} = 1$ supersymmetric nonperturbative vacua of Hořava-Witten M-theory compactified on a torus fibered Calabi-Yau 3-fold Z with first homotopy group $\pi_1(Z) = \mathbb{Z}_2$, having 1) grand unification group $H = E_6$, $SO(10)$, or $SU(5)$, 2) arbitrary net number of generations N_{gen} of chiral fermions in the observable sector, and 3) potentially viable matter Yukawa couplings. The vacua with $H = E_6$, $SO(10)$, or $SU(5)$ correspond to semistable holomorphic vector bundles V_Z over Z having structure group $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ with $n = 3$, 4 or 5, respectively, and generically contain M5-branes in the bulk space.

Construction of Z: We wish to construct a smooth *torus* fibered Calabi-Yau 3-fold Z with $\pi_1(Z) = \mathbb{Z}_2$. To do this, we first construct a smooth *elliptically* fibered Calabi-Yau 3-fold X which admits a freely-acting involution τ_X. We can then construct the quotient manifold $Z = X/\tau_X$.

- **Construction of X:** To construct a smooth elliptically fibered Calabi-Yau 3-fold X which admits a freely-acting involution τ_X,

1. **Choose the base B:** The requirement that $c_1(TX) = 0$ restricts the possible bases [18, 19]. If the base is smooth and preserves only $\mathcal{N} = 1$ supersymmetry in four dimensions, then B is restricted to be a del Pezzo (dP_r, $r = 0, 1, \ldots, 8$), Hirzebruch (F_r, $r \geq 0$), blown-up Hirzebruch, or an Enriques surface ($\mathcal{E}$).

2. **Require two global sections:** To admit a freely-acting involution τ_X, require X to have *two* global sections σ and ξ satisfying
$$\xi + \xi = \sigma. \tag{3.1}$$
Elliptically fibered manifolds can be described in terms of a Weierstrass model. A general elliptic curve can be embedded via a cubic equation into $\mathbb{CP}^2$. Without loss of generality, the equation can be expressed in the Weierstrass form
$$zy^2 = 4x^3 - g_2 z^2 x - g_3 z^3 \tag{3.2}$$
where g_2 and g_3 are general coefficients and (x, y, z) are homogeneous coordinates on $\mathbb{CP}^2$. To define an elliptic fibration over a base B, one needs to specify how the coefficients g_2 and g_3 vary as one moves around the base. In order to have a pair of sections σ and ξ, the Weierstrass polynomial (3.2) must factorize as
$$zy^2 = 4(x - az)(x^2 + azx + bz^2). \tag{3.3}$$
Comparing (3.2) and (3.3), we see that
$$g_2 = 4(a^2 - b), \quad g_3 = 4ab. \tag{3.4}$$
The zero section σ is given by $(x, y, z) = (0, 1, 0)$, and the second section ξ by $(x, y, z) = (a, 0, 1)$.

3. **Blow up singularities:** The elliptic fibers are singular when two roots of the Weierstrass polynomial (3.3) coincide. The set of points in the base over which the fibers are singular is given by the discriminant locus
$$\Delta = 0 \tag{3.5}$$
where
$$\Delta = \Delta_1 \Delta_2^2 \tag{3.6}$$

and

$$\Delta_1 = a^2 - 4b, \quad \Delta_2 = 4(2a^2 + b). \tag{3.7}$$

One can show that there is a curve of singularities over the Δ_2 component of the discriminant curve. To construct the smooth Calabi-Yau 3-fold X, one must blow up this curve of singularities. This is achieved by replacing the singular point of each fiber over $\Delta_2 = 0$ by a sphere $\mathbb{CP}^1$. This is a new curve in the Calabi-Yau 3-fold, which we denote by N. The general elliptic fiber F has now split into two spheres: the new fiber N, plus the proper transform of the singular fiber, which is in the class $F - N$.

- **Choice of involution τ_X:** Construct a freely-acting involution τ_X on X as the composition

$$\tau_X = \alpha \circ t_\xi \tag{3.8}$$

where α is the lift to X of a fibration-preserving involution τ_B on the base B with fixed point set $\mathcal{F}_{\tau_B}$, and

$$t_\xi(x) = x + \xi(x), \quad x \in X \tag{3.9}$$

is an involutive translation of the fibers. To ensure that τ_B preserves the fibration, require

$$\tau_B^*(a) = a, \quad \tau_B^*(b) = b. \tag{3.10}$$

Upon the explicit specification of an involution τ_B with the above properties, the involution α is uniquely determined by the additional requirements that it fix the zero section σ and that it preserve the holomorphic volume form on X.

Note that α leaves fixed the whole fiber above each point in $\mathcal{F}_{\tau_B}$. Since the action of translation on a *smooth* torus acts without fixed points, τ_X will be freely acting provided none of the fibers above $\mathcal{F}_{\tau_B}$ are singular. Thus, require

$$\mathcal{F}_{\tau_B} \cap \{\Delta = 0\} = \emptyset. \tag{3.11}$$

Construction of a vector bundle V_X over X which descends to a vector bundle V_Z over Z:

- $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ **bundle constraints:** We wish to construct (via the spectral cover method [20, 21, 22, 5]) a semi-stable holomorphic vector bundle V_X over X with structure group $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$. To do this, we need to fix a spectral cover C and a line bundle $\mathcal{N}$ over it. The condition that $c_1(V_X) = 0$ implies that the spectral data $(C, \mathcal{N})$ can be written in terms of an effective divisor class η in the base B and coefficients λ and κ_i $(i = 1, \dots, 4\eta \cdot c_1(B))$. Constraints are placed on η , λ , and the κ_i by the condition that

$$c_1(\mathcal{N}) = n\left(\frac{1}{2}+\lambda\right)\sigma + \left(\frac{1}{2}-\lambda\right)\pi_C^*\eta + \left(\frac{1}{2}+n\lambda\right)\pi_C^*c_1(B) + \textstyle\sum_i \kappa_i N_i \tag{3.12}$$

  be an integer class. Various sufficient (but not necessary) constraints can be imposed [5, 6, 7], but most generally, $c_1(\mathcal{N})$ will be an integer class if the constraints

$$q \equiv n\left(\frac{1}{2}+\lambda\right) \in \mathbb{Z} \tag{3.13}$$

$$\left(\frac{1}{2}-\lambda\right)\pi_C^*\eta + \left(\frac{1}{2}+n\lambda\right)\pi_C^*c_1(B) \quad \text{is an integer class} \tag{3.14}$$

$$\kappa_i - \frac{1}{2}m \in \mathbb{Z}, \quad m \in \mathbb{Z} \tag{3.15}$$

  are simultaneously satisfied.

- **Bundle involution conditions:** The bundle V_X over X will descend to a bundle V_Z over Z if V_X is invariant under the involution τ_X. Necessary conditions for V_X to be invariant are given by

$$\tau_B(\eta) = \eta \tag{3.16}$$

$$\textstyle\sum_i \kappa_i = \eta \cdot c_1(B). \tag{3.17}$$

  We note that there may be non-invariant bundles satisfying (3.16) and (3.17); the details of selecting only the invariant bundles are beyond the scope of this paper.

Phenomenological constraints

- N_{gen} **condition:** In the models of interest with V_{Z1} having structure group $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ (with $n = 3$, 4, or 5), the net number of generations (# generations − # antigenerations) N_{gen} of chiral fermions in the observable sector (in the $\mathbf{27} - \overline{\mathbf{27}}$ of E_6, $\mathbf{16} - \overline{\mathbf{16}}$ of $SO(10)$, or $\mathbf{10} + \overline{\mathbf{5}} - (\overline{\mathbf{10}} + \mathbf{5})$ of $SU(5)$) is given by

$$N_{gen} = \frac{1}{2} \int_Z c_3(V_{Z1}). \tag{3.18}$$

Since X is a double cover of Z, it follows that

$$c_3(V_Z) = \frac{1}{2} c_3(V_X). \tag{3.19}$$

$c_3(V_X)$ has been computed by Curio [23] and Andreas [24]:

$$c_3(V_X) = 2\lambda\sigma \wedge \eta \wedge (\eta - nc_1(B)). \tag{3.20}$$

Thus,

$$N_{gen} = \frac{1}{2} \int_B \lambda\eta \wedge (\eta - nc_1(B)) = \frac{1}{2}\lambda\eta \cdot (\eta - nc_1(B)) \tag{3.21}$$

where we have integrated over the fiber and used Poincaré duality.

- **Effectiveness condition:** Anomaly cancellation requires

$$[W_Z] = c_2(TZ) - c_2(V_{Z1}) - c_2(V_{Z2}), \tag{3.22}$$

where $[W_Z]$ is the class associated with non-perturbative M5-branes in the bulk space of the theory. For simplicity, we will take V_{Z2} to be the trivial bundle. Hence, the gauge group E_8 remains unbroken in the hidden sector, $c_2(V_{Z2})$ vanishes, and (3.22) simplifies accordingly. Condition (3.22) can then be pulled back onto X to give

$$[W_X] = c_2(TX) - c_2(V_{X1}). \tag{3.23}$$

The Chern classes appearing in (3.23) have been evaluated to be [5]

$$c_2(TX) = 12\sigma_* c_1 + \left(c_2 + 11c_1^2\right)(F - N) + \left(c_2 - c_1^2\right) N \tag{3.24}$$

$$c_2(V_X) = \sigma_*\eta - \left(f(n) - k^2\right)(F - N) - \left(f(n) - k^2 + \textstyle\sum_i \kappa_i\right) N \quad (3.25)$$

where $c_i \equiv c_i(B)$ and

$$k^2 = \textstyle\sum_i \kappa_i^2 \quad (3.26)$$

$$f(n) = \frac{1}{24}\left(n^3 - n\right) c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right) n\eta \cdot (\eta - nc_1). \quad (3.27)$$

Using these expressions for $c_2(TX)$ and $c_2(V_X)$, (3.23) becomes

$$[W_X] = \sigma_* W_B + c(F - N) + dN \quad (3.28)$$

where

$$c = c_2 + f(n) + 11c_1^2 - k^2 \quad (3.29)$$

$$d = c_2 + f(n) - c_1^2 - k^2 + \textstyle\sum_i \kappa_i \quad (3.30)$$

and

$$W_B = 12c_1(B) - \eta. \quad (3.31)$$

The class $[W_Z]$ must represent a physical holomorphic curve in the Calabi-Yau 3-fold Z since M5-branes are required to wrap around it. Hence $[W_Z]$ must be an effective class, and its pull-back $[W_X]$ is an effective class in the covering 3-fold X. Thus, we require

$$W_B = 12c_1 - \eta \quad \text{is effective in } B \quad (3.32)$$

and

$$c \geq 0, \quad d \geq 0. \quad (3.33)$$

- $\beta_i^{(0)} = 0$ **constraint:** As discussed in [8], to obtain phenomenologically viable matter Yukawa couplings, require vanishing instanton charges, $\beta_i^{(0)}$, on the observable orbifold fixed plane. $\beta_i^{(0)} = 0$ implies that

$$\Omega \equiv c_2(V_{X1}) - \frac{1}{2}c_2(TX) = 0 \quad (3.34)$$

and thus from (3.24) and (3.25)

$$\sigma_*(6c_1 - \eta) + \tilde{c}(F - N) + \tilde{d}N = 0 \quad (3.35)$$

where

$$\tilde{c} = c - \frac{1}{2}c_2 - \frac{11}{2}c_1^2, \tag{3.36}$$

$$\tilde{d} = d - \frac{1}{2}c_2 + \frac{1}{2}c_1^2. \tag{3.37}$$

Thus, we require

$$\eta = 6c_1(B) \tag{3.38}$$

and

$$\tilde{c} = 0 \qquad \tilde{d} = 0. \tag{3.39}$$

- **Stability constraint:** Let $G = SU(n) \subset E_8$ and $G_{\mathbb{C}}$ be the structure group of the vector bundle V_Z. Then the commutant subgroup of G in E_8, denoted by H will be the largest subgroup preserved by V_Z if [25]

$$\eta \geq nc_1(B). \tag{3.40}$$

We note that for the models of interest (which have $n = 3$, 4 and 5), the $\beta_i^{(0)} = 0$ constraint (3.38) ensures that the stability constraint is satisfied.

4 Hirzebruch surfaces

In this section we demonstrate that torus-fibered Calabi-Yau manifolds Z with $\pi_1(Z) = \mathbb{Z}_2$ and Hirzebruch base surfaces do not admit the $H = SO(10)$, $N_{gen} = 3$ vacua with potentially viable matter Yukawa couplings that we seek.

A Hirzebruch surface F_r ($r \geq 0$), is a 2-dimensional complex manifold constructed as a fibration with base $\mathbb{CP}^1$ and fiber $\mathbb{CP}^1$. We denote the class of the base and fiber of F_r by S and E, respectively. Their intersection numbers are

$$S \cdot S = -r \quad S \cdot E = 1 \quad E \cdot E = 0 \tag{4.1}$$

S and E form a basis of the homology class $H_2(F_r, \mathbb{Z})$. This pair has the advantage that it is also the set of generators for the Mori cone. That is, the class

$$\eta = sS + eE \tag{4.2}$$

is effective on F_r for integers s and e if and only if

$$s \geq 0, \quad e \geq 0. \tag{4.3}$$

The Chern classes of F_r are

$$c_1(F_r) = 2S + (r+2)E \tag{4.4}$$
$$c_2(F_r) = 4. \tag{4.5}$$

We will need the result

$$c_1^2(F_r) = 8. \tag{4.6}$$

4.1 $n = 4$ Hirzebruch solutions with $N_{gen} = 3$

We wish to find $n = 4$ Hirzebruch solutions, corresponding to $G = SU(4)$ and $H = SO(10)$, with $N_{gen} = 3$. We begin by imposing the $\beta_i^{(0)} = 0$ constraint (3.38):

$$\eta = 6c_1(F_r). \tag{4.7}$$

With this constraint on η, the second bundle involution condition (3.17) becomes

$$\textstyle\sum_i \kappa_i = \eta \cdot c_1(F_r) = 6c_1^2(F_r) = 48; \quad i = 1, \dots, 192 \tag{4.8}$$

and the N_{gen} condition (3.21), with $N_{gen} = 3$, becomes

$$3 = N_{gen} = \frac{1}{2}\lambda 6(6-n)c_1^2(F_r). \tag{4.9}$$

For $n = 4$, we obtain

$$\lambda = \frac{1}{16}. \tag{4.10}$$

Plugging this value for λ along with $n = 4$ into (3.13), we see that the $G_{\mathbb{C}} = SU(4)_{\mathbb{C}}$ bundle constraints cannot be satisfied and hence there are no $n = 4$ Hirzebruch solutions with $N_{gen} = 3$. It is interesting to note that the requirement of vanishing instanton charges on the observable orbifold plane rules out the $n = 4$ Hirzebruch solutions presented in [7].

5 Del Pezzo surfaces

In this section we demonstrate the existence of a torus fibered Calabi-Yau 3-fold Z with $\pi_1(Z) = \mathbb{Z}_2$ and del Pezzo base surface dP_7 which admits $H = SO(10)$, $N_{gen} = 3$ vacua with potentially viable matter Yukawa couplings.

A del Pezzo surface dP_r ($r = 0, 1, \ldots, 8$), is a 2-dimensional complex manifold constructed from complex projective space $\mathbb{CP}^2$ by blowing up r points. A basis of $H_2(dP_r, \mathbb{Z})$ composed of effective classes is given by the hyperplane class l and r exceptional divisors E_i, $i = 1, \ldots, r$. Their intersections are

$$l \cdot l = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad E_i \cdot l = 0. \tag{5.1}$$

The Chern classes are given by

$$c_1(dP_r) = 3l - \sum_{i=1}^{r} E_i \tag{5.2}$$

$$c_2(dP_r) = 3 + r. \tag{5.3}$$

We will need the result

$$c_1^2(dP_r) = 9 - r. \tag{5.4}$$

5.1 $n = 4$ del Pezzo solutions with $N_{gen} = 3$

We wish to find $n = 4$ del Pezzo solutions, corresponding to $G = SU(4)$ and $H = SO(10)$, with $N_{gen} = 3$. We begin by imposing the $\beta_i^{(0)} = 0$ constraint (3.38):

$$\eta = 6c_1(dP_r) \tag{5.5}$$

With this constraint on η, the second bundle involution condition (3.17) becomes

$$\textstyle\sum_i \kappa_i = \eta \cdot c_1(dP_r) = 6c_1^2(dP_r) = 6(9 - r); \quad i = 1, \ldots, 24(9 - r) \tag{5.6}$$

and the N_{gen} condition (3.21), with $N_{gen} = 3$, becomes

$$3 = N_{gen} = \frac{1}{2}\lambda 6(6 - n)c_1^2(dP_r). \tag{5.7}$$

r	0	1	2	3	4	5	6	7	8
λ	1/18	1/16	1/14	1/12	1/10	1/8	1/6	1/4	1/2
q	20/9	9/4	16/7	7/3	12/5	5/2	8/3	3	4

Table 5.1: The second line contains the $n = 4$ del Pezzo dP_r values for λ given by (5.8). $q \equiv n\left(\frac{1}{2} + \lambda\right)$ must be an integer for the $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ bundle constraint (3.13) to be satisfied.

For $n = 4$, we obtain

$$\lambda = \frac{1}{2(9-r)}. \tag{5.8}$$

The values of λ given by (5.8) for each r are given in Table 5.1. In this table, the quantity $q \equiv n\left(\frac{1}{2} + \lambda\right)$ must be an integer for the $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ bundle constraint (3.13) to be satisfied. From the table, we see that when $n = 4$, this constraint can be satisfied only for $r = 7$ or $r = 8$. Thus, we can exclude the dP_r $(r = 0, \dots, 6)$ surfaces from consideration.

We now try to satisfy the second $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ bundle constraint (3.14). Using (5.5) in (3.14) gives

$$\left[\frac{7}{2} + \lambda(n-6)\right] \pi_C^* c_1(dP_r) \quad \text{is an integer class.} \tag{5.9}$$

Thus, (5.9) is satisfied if

$$p \equiv \frac{7}{2} + \lambda(n-6) \in \mathbb{Z}. \tag{5.10}$$

For the $r = 7$ and $r = 8$ del Pezzo surfaces, we find

$$p(\lambda = 1/4, n = 4) = 3 \tag{5.11}$$
$$p(\lambda = 1/2, n = 4) = 5/2 \notin \mathbb{Z} \tag{5.12}$$

respectively. Thus, the $r = 8$ del Pezzo surface is excluded, and the only remaining possibility is

$$\lambda(r = 7, n = 4) = 1/4. \tag{5.13}$$

We note that this value of λ would not be permitted if the sufficient (but not necessary) $G_{\mathbb{C}} = SU(n)_{\mathbb{C}}$ bundle constraints discussed in [7] had been imposed. Using (5.13) and $\eta = 6c_1(dP_7)$ in (3.29) and (3.30) gives

$$c = 46 - k^2 \tag{5.14}$$
$$d = 34 - k^2. \tag{5.15}$$

Imposing the effectiveness conditions $c \geq 0$ and $d \geq 0$ gives

$$k^2 \equiv \textstyle\sum_i \kappa_i^2 \leq 34. \tag{5.16}$$

Furthermore, $\eta = 6c_1(dP_7)$ implies that

$$W_B = 12c_1(dP_7) - \eta = 6c_1(dP_7) \tag{5.17}$$

which means that W_B is indeed effective.

Using $c_1^2(dP_7) = 2$, $c_2(dP_7) = 10$ and our results $c = 46 - k^2$, $d = 34 - k^2$, (3.36) and (3.37) become

$$\tilde{c} = \tilde{d} = 30 - k^2. \tag{5.18}$$

Imposing the $\beta_i^{(0)} = 0$ constraints $\tilde{c} = \tilde{d} = 0$ gives

$$k^2 \equiv \textstyle\sum_i \kappa_i^2 = 30 \tag{5.19}$$

which is consistent with (5.16). One needs therefore a set of κ_i which simultaneously satisfy

$$\textstyle\sum_i \kappa_i = 6c_1^2(dP_7) = 12; \quad i = 1, \ldots, 48 \tag{5.20}$$

and (5.19) for κ_i obeying the bundle constraint (3.15). An example of such κ_i is

$$\kappa_1 = \kappa_2 = \kappa_3 = 2, \quad \kappa_4 = \kappa_5 = 3, \quad \text{all other } \kappa_i = 0. \tag{5.21}$$

Thus, $n = 4$ dP_7 solutions with $N_{gen} = 3$ exist whenever the constraints (3.10), (3.11), and (3.16) on the involution τ_X are satisfied.

6 Toward nonperturbative top quark mass

In this section we discuss how the 11-dimensional framework of Hořava-Witten M-theory may be used to extend the perturbative calculation of the top quark Yukawa coupling in the realistic free-fermionic models to the non-perturbative regime.

Let us recall that in the free-fermionic heterotic string formalism [26, 27], a model is specified in terms of a set of boundary condition basis vectors and one-loop GSO projection coefficients. The realistic free-fermionic models of interest here are constructed in two stages. The first stage corresponds to the NAHE set of boundary condition basis vectors $\{\mathbf{1}, S, b_1, b_2, b_3\}$ [28]. At the second stage, we add to the NAHE set three boundary condition basis vectors, typically denoted by $\{\alpha, \beta, \gamma\}$. The gauge group at the level of the NAHE set is $SO(6)^3 \times SO(10) \times E_8$, which is broken to $SO(4)^3 \times U(1)^3 \times SO(10) \times SO(16)$ by the vector 2γ. Alternatively, we can start with an extended NAHE set $\{\mathbf{1}, S, \xi_1, \xi_2, b_1, b_2\}$ [9], with $\xi_1 = \mathbf{1} + b_1 + b_2 + b_3$. The set $\{\mathbf{1}, S, \xi_1, \xi_2\}$ produces a toroidal Narain model with $SO(12) \times E_8 \times E_8$ or $SO(12) \times SO(16) \times SO(16)$ gauge group for appropriate choices of the GSO phase $c\binom{\xi_1}{\xi_2}$. The basis vectors b_1 and b_2 then break $SO(12) \to SO(4)^3$, and either $E_8 \times E_8 \to E_6 \times U(1)^2 \times E_8$ or $SO(16) \times SO(16) \to SO(10) \times U(1)^3 \times SO(16)$. The vectors b_1 and b_2 correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold modding. The three vectors b_1, b_2, and b_3 correspond to the three twisted sectors of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, each producing eight generations in the $\mathbf{27}$ representation of E_6 or $\mathbf{16}$ representation of $SO(10)$. In the case of E_6, the untwisted sector produces an additional $3 \times (\mathbf{27} + \overline{\mathbf{27}})$, whereas in the $SO(10)$ model it produces $3 \times (\mathbf{10} + \overline{\mathbf{10}})$. Therefore, the Calabi-Yau 3-fold which corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at the free-fermionic point in the Narain moduli space has $(h^{(1,1)}, h^{(2,1)}) = (27, 3)$.

This basic structure underlies all realistic free fermionic models. In the second stage of the contruction the $SO(10)$ symmetry is broken to one of its subgroups and the number of generations is reduced to three, one from each of the twisted sectors b_1, b_2 or b_3. The top quark is identified with the leading mass state. The Yukawa coupling of this mass state is obtained at the cubic level of the superpotential and is a coupling between states from the twisted-twisted-untwisted sectors. For example, in the standard-like models the relevant coupling is $t_1^c Q_1 \bar{h}_1$, where t_1^c and Q_1 are respectively the quark $SU(2)$ singlet and doublet from the sector b_1, and $\bar{h}_1$ is the untwisted Higgs. Thus, one can calculate this coupling in the full three generation model or at the level of the (51,3) or (27,3) $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, and as a $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ $SO(10)$ coupling, or a $\mathbf{27}^3$ E_6 coupling. As long as the moduli are fixed at the free-fermionic point, the numerical results will be identical. While we do not know the precise geometrical realization of the three generation models, the geometry of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is more readily identified. Thus, to extend the calculation of the top quark Yukawa coupling in the realistic free-fermionic models to the nonperturbative regime, one can compactify Hořava-Witten M-theory on a Calabi-Yau 3-fold which corresponds to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. $SU(n)_{\mathbb{C}}$ vector bundles with $n = 3$ or $n = 4$ can be chosen, corresponding to the E_6 or $SO(10)$ grand unification group, respectively. The nonperturbative top quark Yukawa coupling at the grand unification scale M_G is then computed, at least in principle, using (2.21). However, this calculation may require modifications to the rules presented in Section 3 in the sense that we now discuss.

Let X_1 be the Calabi-Yau 3-fold which corresponds to the $(51, 3)$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. As discussed in [7], this manifold has the structure of the manifold X described in Section 3. X_1 can be realized as a singular limit of the $(3, 243)$ elliptically-fibered Calabi-Yau 3-fold X_1' with base $\mathbb{CP}^1 \times \mathbb{CP}^1$ [29]. We can represent the fibers of X_1' in Weierstrass form

$$y^2 = x^3 + f_8(w, \tilde{w})z^4 x + g_{12}(w, \tilde{w})z^6 \tag{6.1}$$

where $w, \tilde{w}$ are inhomogeneous coordinates of the respective $\mathbb{CP}^1$. Making the choices

$$f_8 = \eta - 3h^2, \quad g_{12} = h(\eta - 2h^2) \tag{6.2}$$

where

$$h = K \prod_{i,j=1}^{4} (w - w_i)(\tilde{w} - \tilde{w}_j), \quad \eta = C \prod_{i,j=1}^{4} (w - w_i)^2 (\tilde{w} - \tilde{w}_j)^2, \tag{6.3}$$

we have a D_4 singular fiber as we approach any of the $w = w_i$ (or $\tilde{w} = \tilde{w}_j$). These D_4 singularities intersect in 16 points, $(w_i, \tilde{w}_j)$, $i, j = 1, \ldots, 4$ in the base. To obtain the $(51, 3)$, resolving the singular fibers is not enough. One must also blow up the base once at each $(w_i, \tilde{w}_j)$, $i, j = 1, \ldots, 4$. This blow-up procedure differs from the prescription in Section 3. Thus, a detailed nonperturbative extension of the top quark Yukawa coupling calculation in the realistic free fermionic models may require modifications to the rules presented in Section 3. We remark that the nonperturbative calculation of the remaining matter Yukawa couplings [30] requires more detailed knowledge of the geometry of the three generation free-fermionic models.

7 Conclusions

Using the rules presented in Section 3, we have searched for $\mathcal{N}=1$ supersymmetric nonperturbative vacua of Hořava Witten M-theory compactified on a torus-fibered Calabi-Yau 3-fold Z with $\pi_1(Z)=\mathbb{Z}_2$ having 1) $SO(10)$ grand unification group, 2) net number of generations $N_{gen}=3$ of chiral fermions in the observable sector and 3) potentially viable matter Yukawa couplings. These vacua correspond to semistable holomorphic vector bundles V_Z over Z having structure group $SU(4)_{\mathbb{C}}$, and generically contain M5-branes in the bulk space. We have demonstrated that torus fibered Calabi-Yau 3-folds Z with $\pi_1(Z)=\mathbb{Z}_2$ and Hirzebruch base surfaces do not admit such vacua, but those with a del Pezzo dP_7 base surface do. The extension of the top quark Yukawa coupling calculation in the realistic free-fermionic models to the non-perturbative regime was discussed. It appears that a detailed analysis will require modifications to the rules presented in Section 3. We hope to make these modifications and perform a detailed analysis in a future publication.

Acknowledgements

We would like to thank Jose Isidro for discussions during the initial stages of this project. A.E.F. is supported in part by PPARC.

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